1. Derive the jump conditions for $u''(x) = f(x) + C\delta(x-\alpha) + \bar{C}\delta'(x-\alpha)$, $x \in (a,\ b)$, $\alpha \in (a,\ b)$, $f(x) \in C(a,\ b)$. **Hint:** Multiply the equation by testing function $\phi(x) \in C_0^2(a,\ b)$ and then integrate.

2. Explore the problem of $u''(x) = f(x) + C\delta(x)$, $x \in (0,\ 1)$, $u(1) = 0$.

3. Explain uniform, adaptive, body fitted meshes for $u''(x) = f(x) + C\delta(x-\alpha) + \bar{C}\delta'(x-\alpha)$, $x \in (a,\ b)$, $\alpha \in (a,\ b)$, $f(x) \in C(a,\ b)$. What are the advantage and dis-advantages?

4. Find the Green function $G''(x,\bar{x}) = \delta(x-\bar{x})$, $x \in (0,\ 1)$, $\bar{x} \in (0,\ 1)$, $\alpha G(0,\bar{x}) + \beta G'(0,\bar{x}) = 0$, $\beta \neq 0$. Show that the finite difference method using the ghost point method has second order convergence in the infinity norm. Also show that the solution to the discrete delta function has the same value as that of $G''(x,\bar{x})$ at all the grid points.

5. For an interface problem, $(\beta u')' - \sigma u = f(x) + C\delta(x-\alpha) + \bar{C}\delta'(x-\alpha)$, also assume that $\beta(x)$ has a finite jump at $x = \alpha$. Explain the following:

   (a) Uniform mesh, fitted mesh, adaptive mesh.

   (b) Smoothing method for the coefficient, harmonic averaging technique, Peskin's IB method.

   (c) Use the IIM to solve the problem and compare with other methods.

6. Use Peskin's IB and IIM methods to solve $(\beta u')' = C\delta(x-\alpha)$, $x \in (0,\ 1)$, $\alpha \in (0,\ 1)$, $u'(0) = 0$, $u(1) = 0$..

   (a) Find the true solution.

   (b) Find the local truncation errors of IB and IIM. For IB, consider the discrete hat delta function, and discrete cosine delta function. IS IB method consistent, convergent?

   (c) Plot the true and computed solution; and the errors.

7. Consider the following finite difference scheme for solving the two point boundary value problem $u''(x) = f(x)$, $a < x < b$, $u(a) = u_a$ and $u(b) = u_b$.

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} = f(x_i), \quad i = 2, 3, \cdots, n-1. \tag{1}$$

   where $x_i = a + ih$, $i = 0, 1, \cdots, n$, $h = (b-a)/n$. At $i = 1$, the finite difference scheme is

$$\frac{U_1 - 2U_2 + U_3}{h^2} = f(x_1). \tag{2}$$

   (a) Find the local truncation errors of the finite difference scheme. Is this scheme consistent?

   (b) Does this scheme converge? Justify your answer.

8. Program the central finite difference method for the self-adjoint BVP

$$\begin{aligned}(\beta(x)u')' - \gamma(x)u(x) &= f(x), \quad 0 < x < 1, \\ u(0) = u_a, &\qquad au(1) + bu'(1) = c.\end{aligned}$$

using a uniform grid and the central difference scheme

$$\frac{\beta_{i+\frac{1}{2}}(U_{i+1} - U_i)/h - \beta_{i-\frac{1}{2}}(U_i - U_{i-1})/h}{h} - \gamma(x_i)U_i = f(x_i). \tag{3}$$

Test your code for the case where

$$\beta(x) = 1 + x^2, \quad \gamma(x) = x, \quad a = 2, \quad b = -3, \tag{4}$$

and the rest of functions or parameters are determined from the exact solution

$$u(x) = e^{-x}(x-1)^2. \tag{5}$$

Plot the computed solution and the exact solution, and the errors for a particular grid, say $n = 80$. Do the grid refinement analysis to determine the order of accuracy of the global solution. Also try to answer the following questions:

- Can your code handle the case when $a = 0$ or $b = 0$?
- If we use the central difference scheme for the equivalent differential equation

$$\beta u'' + \beta' u' - \gamma u = f, \tag{6}$$

what are the advantages or disadvantages?

9. Consider the finite difference scheme for the 1D steady state *convection-diffusion* equation

$$\epsilon u'' - u' = -1, \quad 0 < x < 1 \tag{7}$$

$$u(0) = 1, \qquad u(1) = 3. \tag{8}$$

(a) Verify the exact solution is

$$u(x) = 1 + x + \left(\frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}\right). \tag{9}$$

(b) Compare the following two finite difference methods for $\epsilon = 0.3$, 0.1, 0.05, and 0.0005,
(1): Central difference scheme:

$$\epsilon\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_{i+1} - U_{i-1}}{2h} = -1. \tag{10}$$

(2): Central-upwind difference scheme:

$$\epsilon\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} - \frac{U_i - U_{i-1}}{h} = -1. \tag{11}$$

Do grid refinement analysis for each case to determine the order of accuracy. Plot the computed solution and the exact solution for $h = 0.1$, $h = 1/25$, and $h = 0.01$. You can use Matlab command *subplot* to put several graphs together.

(c) From you observation, give your opinion to see which method is better.