Fourth order compact scheme for 1D BVP

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Abstract

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1 Fourth order compact scheme for 1D Sturm-Liouville BVP

Consider

$$(p'u')' + Ku = f, \qquad x_l < x < x_r, \qquad u(x_l) = u_a, \quad u(x_l) = u_b. \tag{1}$$

We assume that $p(x) \geq p_0 > 0$.

1.1 Constant p

If this case, we simply take $p = 1$ and assume that $K = 0$. We have the following lemma for a three-point finite difference formula. Given a grid point x_i. Assume that $x_i - x_{i-1} = h_1$ and $x_{i+1} - x_i = h_2$. We design the following compact finite difference scheme,

$$\sum_{k=-1}^{1} \alpha_k U_{i+k} = \sum_{k=-1}^{1} \beta_k f_{i+k}. \tag{2}$$

The coefficients are given as

$$\begin{aligned} \alpha_1 &= \frac{2}{h_1(h_1+h_2)}, \qquad \alpha_3 = \frac{2}{h_2(h_1+h_2)}, \qquad \alpha_2 = -\left(\alpha_l + \alpha_r\right) = -\frac{2}{h_1 h_2}, \\ \beta_1 &= \frac{h_1^2 - h_2^2 + hh_2}{6h_1(h_1+h_2)}, \qquad \beta_3 = \frac{-h_1^2 + h_2^2 + hh_2}{6h_2(h_1+h_2)}, \qquad \beta_2 = \frac{h_2^2 + h^2 + 3h_1h_2}{6h_1h_2}. \end{aligned} \tag{3}$$

Then, the finite difference scheme is exact if $u(x)$ is any polynomial of degree 4 and the local truncation error T_i

$$\sum_k \beta_k = 1, \qquad |T_i| \leq C \max\left\{h_1^3, h_2^3\right\} \tag{4}$$

and $|T_i| \leq Ch^4$ if $h_1 = h_2 = h$. We can see the 'symmetry' between the left and right grid points as expected. Note that when $h_1 = h_2$ we have the standard fourth-order compact formula. The finite

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difference coefficients satisfy the sign property needed for an M-matrix conditions. When $K \neq 0$, we can treat Ku as a source term like f, that is, we add the

$$\beta_1 U_{i-1} + \beta_2 U_i + \beta_3 U_{i+1}, \tag{5}$$

term to the finite difference equation at the right hand side.

Thus, we will have fourth order accurate and stable finite difference method if a uniform mesh is used. The non-uniform mesh can be used for a few isolated grid points without affecting global fourth order accuracy.

1.2 Variable $p(x)$

For simplicity of the discussion we assume that $K = 0$ and a uniform mesh. We also need $p'(x)$ and $p''(x)$ or third order approximations. Define

$$\hat{p}(x) = p - \frac{h^2}{12}\frac{(p')^2}{p} + \frac{h^2}{24}p'' \tag{6}$$

which depends on the mesh size h. The fourth order FD equation then is

$$\frac{\hat{p}_{i+\frac{1}{2}}\, U_{i+1} - \left(\hat{p}_{i+\frac{1}{2}} + \hat{p}_{i-\frac{1}{2}}\right) U_i + \hat{p}_{i-\frac{1}{2}}\, U_{i-1}}{h^2} = F_i, \tag{7}$$

where

$$F_i = \frac{f_{i-1} + 10 f_i + f_{i+1}}{12} - \frac{h^2}{12}\frac{\left(\frac{p' f}{p}\right)(x_{i+\frac{1}{2}}) - \left(\frac{p' f}{p}\right)(x_{i-\frac{1}{2}})}{2h}. \tag{8}$$

Note that the first term is the same as that when $p = 1$. The proof is based on an important lemma.

Lemma 1 *Assume that $u(x) \in C^4$, then*

$$\frac{u(x+h/2) - u(x-h/2)}{h} = u'(x) + \frac{h^2}{24}u'''(x) + O(h^4). \tag{9}$$

Derivation of the FD equation: From $(pu')' = f$ and the lemma, we have

$$\begin{aligned}
\frac{p_{i+\frac{1}{2}} u'(x_{i+\frac{1}{2}}) - p_{i-\frac{1}{2}} u'(x_{i-\frac{1}{2}})}{h} &= (pu')'(x_i) + \frac{h^2}{24}(pu')'''(x_i) + O(h^4) \\
&= f(x_i) + \frac{h^2}{24} f''(x_i) + O(h^4).
\end{aligned}$$

We further apply the lemma to obtain

$$\begin{aligned}
\frac{p_{i+\frac{1}{2}} \frac{u(x_{i+1}) - u(x_i)}{h} - p_{i-\frac{1}{2}} \frac{u(x_i) - u(x_{i-1})}{h}}{h} &= f(x_i) + \frac{h^2}{24} f''(x_i) + O(h^4) \\
&\quad + \frac{h}{24}\left(p_{i+\frac{1}{2}} u'''(x_{i+\frac{1}{2}}) - p_{i-\frac{1}{2}} u'''(x_{i-\frac{1}{2}})\right) + O(h^4) \\
&= f(x_i) + \frac{h^2}{24} f''(x_i) + \frac{h^2}{24}(pu''')'(x_i) + O(h^4).
\end{aligned}$$

Thus, the key is to have a second order approximation to the $(pu''')(x_i)$ term. From the ODE $(pu')' = f$, we know that

$$u'' = \frac{f - p'u'}{p}. \tag{10}$$

We differentiate $(pu')' = f$ once more and expand to get

$$pu''' + 2p'u'' + p''u' = f'. \tag{11}$$

From the above, we solve for u''' and plug the expression of u'' to get

$$u''' = \frac{f'}{p} - \frac{2p'(f - p'u')}{p^2} - \frac{p''u'}{p}. \tag{12}$$

Thus, we have

$$\frac{h^2}{24}(pu''')' = \frac{h^2}{24} f'' - \frac{h^2}{24}\left(\frac{2p'f}{p}\right)' + \frac{h^2}{24}\left(\left(\frac{2(p')^2}{p} - p''\right)u'\right)'. \tag{13}$$

Finally we apply the central finite difference formula for the above three terms and note that the last term is a self-adjoint second order finite difference operator that can be combined with the $(pu')'$ term, then we get the designed finite difference equation and the local truncation error.

Example 1 *Let the true solution be* $u(x) = \sin(kx)$, $0 < x < \pi$, $p(x) = 1 + x^2$. *The source term is* $f(x) = 2x\cos(kx) - k^2(1 + x^2)\sin(kx)$. *Table 1 is a grid refinement analysis when* $k = 3$.

Table 1: A grid refinement analysis of the third order compact finite element method for Example 1 with $k = 5\pi$.

N	$\|E\|_\infty$	Order
16	7.2920e-04	
32	4.4976e-05	4.0191
64	2.8017e-06	4.0048
128	1.7496e-07	4.0012
256	1.0938e-08	3.9996
512	6.8355e-10	4.0002
1024	4.1655e-11	4.0365
2056	2.7427e-12	3.9248

2 A new third order method based on a posterior error analysis

Now we consider $u''(x) = -f(x)/\beta$. It is well-known that the finite element solution is the exact at nodal points for the special case. Thus the interpolation function is the same as the finite element solution,

$$u_h^I(x) = \sum_{j=1}^{M} u(x_j)\phi_j(x) \;\; = \sum_{j=1}^{M} c_j^*\phi_j(x) = u_h(x), \tag{14}$$

From the classical interpolation theory, see for example, [], we know that on an element $e_k = [x_k, x_{k+1}]$, the following is true

$$\begin{aligned} u(x) &= u_h^I(x) + \frac{1}{2}\big(x - x_k\big)\big(x - x_{k+1}\big)u''(x) + O(h^3) \\ &= u_h(x) - \frac{1}{2}\big(x - x_k\big)\big(x - x_{k+1}\big)\,\frac{f(x)}{\beta} + O(h^3). \end{aligned} \tag{15}$$

Thus we obtain a third method using a posterior error estimate with a simple correction term that can be easily calculated.

Theorem 1 *Let $u(x) \in H^2(x_l, x_r)$, $f(x) \in L^2(x_l, x_r)$, and $u_h(x)$ be the finite element solution obtained using the P_1 finite element space, then*

$$u_h^{new}(x) = u_h(x) - \frac{1}{2\beta}\big(x - x_k\big)\big(x - x_{k+1}\big)\, f(x), \qquad x_k < x < x_{k+1}, \tag{16}$$

is a third order approximation to the true solution $u(x)$ and we have the following error estimates

$$\|u - u_h^{new}\|_{L^2} \le Ch^3, \qquad \|u - u_h^{new}\|_{H^1} \le Ch^2, \qquad \|u - u_h^{new}\|_e \le Ch^2. \tag{17}$$

Note also that $u_h^{new}(x_k) = u(x_k)$ for all k's. That is, the solution has one more order accuracy.